ON A MULTIDIMENTIONAL BROWNIAN MOTION WITH PARTLY REFLECTING MEMBRANE ON A HYPERPLANE

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Abstract. A multidimensional Brownian motion with partial reflection on a hyperplane S in the direction $qN+\alpha$, where N is the conormal vector to the hyperplane and $q\in[-1,1], \alpha\in S$ are given parametres, is constructed and this construction is based on both analytic and probabilistic approaches. The joint distribution of $d-$dimensional analogy to skew Brownian motion and its local time on the hyperplane is obtained.

§1. Introduction.

In this paper we construct a generalized diffusion process in a d-dimensional Euclidean space $\Re^d$ for which the diffusion matrix B is constant and the drift vector is equal to $(qB\nu+\alpha)\delta_S(x)$ where $\nu\in\Re^d$ is a given unit vector, S is the hyperplane in $\Re^d$ orthogonal to ν, the parameters $q\in[-1,1]$ and $\alpha\in S$ are given and $\delta_S(x)$ is a generalized function that is determined by the relation

$$\int_{\Re^d}\varphi(y)\delta_S(y)dy=\int_S\varphi(y)d\sigma$$

valid for any test function φ.

This problem in a more general case (namely, the parameters α, q are some functions of $x\in S$ and the diffusion matrix B is a function of $x\in\Re^d$) was considered by Kopytko B.I. [1] and solved by an analytic method. More simple case (B is an identical operator, α, q are some constants) was considered by Kopytko B.I., Portenko N.I. [2] with use of an analytic method too.

In this work a probabilistic approach to this problem will be proposed. We construct the process desired as a solution of the following stochastic differential equation

$$dx(t)=(qB\nu+\alpha)\delta_S(x(t))dt+B^{1/2}dw(t)$$

where $B^{1/2}$ is the positive square root from the operator B and $w(t)$ is a given Wiener process in $\Re^d$. If $\alpha=0$ then the solution of this equation is known. Let us denote it by $\tilde{x}(t)$. The idea is to add the process $\alpha\int_0^t\delta_S(\tilde{x}(\tau))d\tau$ to the projection of $\tilde{x}(t)$ on S.

The paper consists of two sections. In Section 2 we describe the analytic method. It is similar to that of [2]. The probabilistic approach is presented in Section 3.

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§2. The Analytic Method.

Let $\nu \in \Re^d$ be a fixed vector with $\|\nu\| = 1$ and S be the hyperplane in $\Re^d$ orthogonal to ν

$$S = \left\{x \in \Re^d | (x, \nu) = 0\right\}$$

And let $x(t)$ be a continuous Markov process in $\Re^d$ with transition probability density $g_0(t, x, y)$

$$g_0(t, x, y) = [det(2\pi tB)]^{-1/2}\, exp\left\{-\frac{1}{2t}(B^{-1}(y-x), y-x)\right\}, t > 0, x \in \Re^d, y \in \Re^d$$

where B is the given positive definite symmetric operator in $\Re^d$.

Let $q \in [-1, 1], \alpha \in S$ be given. We are looking for a function $u(t, x, \varphi)$ defined for $t > 0, x \in \Re^d, \varphi \in C_b(\Re^d)$ such that

(1) $u(t, x, \varphi)$ satisfies the heat equation in the domain $t > 0, x \notin S$

$$\frac{\partial u}{\partial t} = \frac{1}{2} Tr(BD^2 u)$$

If we fix a coordinates in $\Re^d$ then $D^2 u = (\frac{\partial^2 u}{\partial x_i \partial x_j})_{i,j=1}^d$ and this equation can be written as follows

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} b_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}$$

where $B = (b_{ij})_{i,j=1}^d$.

(2) $u(t, x, \varphi)$ satisfies the initial condition

$$\lim_{t\downarrow 0} u(t, x, \varphi) = \varphi(x), x \in \Re^d, \varphi \in C_b(\Re^d)$$

(3) For $x \in S$ and $t > 0$ the relation

$$u(t, x+, \varphi) = u(t, x-, \varphi)$$

holds, where $u(t, x\pm, \varphi) = \lim_{\varepsilon\downarrow 0} u(t, x \pm \varepsilon\nu, \varphi)$

(4) The relation

$$\frac{1+q}{2}\frac{\partial u(t, x+, \varphi)}{\partial N} - \frac{1-q}{2}\frac{\partial u(t, x-, \varphi)}{\partial N} + (\alpha, \bigtriangledown u) = 0,$$

is valid for $x \in S$ and $t > 0$, where $\bigtriangledown u$ is gradient of $u(t, x, \varphi)$, N is the conormal vector to $S, N = B\nu$.

Let us denote by $x_\nu(t)$ the projection of $x(t)$ on ν, by π_S the operator of orthogonal projection on S such that $x^S(t) = \pi_S x(t)$ is the projection of $x(t)$ on S. If ν is not an eigen vector of the operator B then $x_\nu(t)$ and $x^S(t)$ are not independent processes and we can not represent $g_0(t, x, y)$ as a product of $x_\nu(t)$ and $x^S(t)$ densities. But we may consider the following decomposition of $g_0(t, x, y)$

$$g_0(t, x, y) = \frac{1}{\sqrt{2\pi t\sigma^2}} \exp\left\{-\frac{(y_\nu - x_\nu)^2}{2t\sigma^2}\right\} \times$$

$$\times [det(2\pi t B_S)]^{-1/2} \, exp\left\{-\frac{1}{2t}(B_S^{-1}(y_S - x_S - \frac{y_\nu - x_\nu}{\sigma^2}b), y_S - x_S - \frac{y_\nu - x_\nu}{\sigma^2}b)\right\}$$

where $b = \pi_S B\nu, B_S = (\pi_S B^{-1}\pi_S)^{-1}, \sigma^2 = (B\nu, \nu)$.

The first factor of this decomposition is the density of the process $x_\nu(t)$, the second one is the conditional density of the process $x^S(t)$ under the condition $x_\nu(t) = y_\nu$. Denoting by $g^S(t, x, y)$ the second factor we may write

$$g_0(t, x, y) = \frac{1}{\sqrt{2\pi t\sigma^2}} \exp\left\{-\frac{(y_\nu - x_\nu)^2}{2t\sigma^2}\right\} g^S(t, x, y), t > 0, x \in \Re^d, y \in \Re^d$$

The problem $(1) - (4)$ may be solved in the following manner. Let us fix a system of coordinates so that $\nu = e_d$ and make use of the Fourier transformation with respect to $x_1, ..., x_{d-1}$ and the Laplace transformation with respect to t to the function $u(t, x, \varphi)$. It will reduce this problem to solving a pair of second order differential equations with constant coefficients. Then free coefficients may be found from boundary conditions $(3), (4)$. Returning to the original function we get the following assertion.

Theorem 1. *If $|q| \le 1$ then there exists a continuous Markov process in $\Re^d$ with transition probability density $G(t, x, y)$ such that*

$$u(t, x, \varphi) = \int_{\Re^d} \varphi(y) G(t, x, y) dy, t > 0, x \in \Re^d, \varphi \in C_b(\Re^d)$$

where

$$G(t, x, y) = \frac{1}{\sqrt{2\pi t\sigma^2}} \left[\exp\left\{-\frac{(y_\nu - x_\nu)^2}{2t\sigma^2}\right\} - \exp\left\{-\frac{(|y_\nu| + |x_\nu|)^2}{2t\sigma^2}\right\}\right] g^S(t, x, y) +$$

$$+ \int_0^\infty (q \,\mathrm{sign}\, y_\nu + 1) \frac{\sigma^2\theta + |x_\nu| + |y_\nu|}{\sqrt{2\pi t^3\sigma^2}} \exp\left\{-\frac{(\sigma^2\theta + |x_\nu| + |y_\nu|)^2}{2t\sigma^2}\right\} g^S(t, x + \alpha\theta, y) d\theta$$

Remark. The condition $|q| \le 1$ provides that $G(t, x, y)$ is positive.

§3. The Probabilistic Method.

Let $w(t)$ be a given d-dimensional Wiener process, B be a symmetric positive definite operator in $\Re^d$ and parameters $q \in [-1, 1], \alpha \in S$ be given. Consider the stochastic differential equation

$$dx(t) = (qB\nu + \alpha)\delta_S(x(t))dt + B^{1/2}dw(t) \tag{1}$$

where $\delta_S(x)$ is the generalized function on $\Re^d$ defined above.

When $\alpha = 0$ then the solution of the equation (1) is known. It may be called a multidimensional analogy to skew Brownian motion. Let us denote it by $\tilde{x}(t)$. The density of $\tilde{x}(t)$ equals to

$$\tilde{g}(t, x, y) = \frac{1}{\sqrt{2\pi t\sigma^2}} \left[\exp\left\{-\frac{y_\nu - x_\nu)^2}{2t\sigma^2}\right\} + q \,\mathrm{sign}\, y_\nu \exp\left\{-\frac{(|y_\nu| + |x_\nu|)^2}{2t\sigma^2}\right\}\right] g^S(t, x, y)$$

where $t > 0, x \in \Re^d, y \in \Re^d$

The equation (1) means that we have to find the process $x(t)$ that satisfies the following properties. The projection of $x(t)$ on ν is skew Brownian motion, the projection of $x(t)$ on S is the Gaussian process in S (with mean $x^S + \frac{y_\nu - x_\nu}{\sigma^2} b$ and correlation operator tB_S) plus the process $\alpha\eta_t$, where η_t is the functional of the process $x(t)$ such that

$$\eta_t = \int_0^t \delta_S(x(\tau))d\tau$$

Remark. Actually η_t depends only on $x_\nu(t)$. Therefore we may write $\eta_t = \int_0^t \delta_S(\tilde{x}(\tau))d\tau$

The functional η_t is a nonnegative continuous homogeneous additive functional, it increases at those instants of time for which the process $\tilde{x}(t)$ hits the hyperplane S.

To solve the equation (1) we have to find the joint distribution of $\tilde{x}(t)$ and η_t.

Lemma 1. *The joint distribution of $\tilde{x}(t)$ and η_t has the form*

$$\mathbf{P}_x\left\{\tilde{x}(t) \in dy, \eta_t \in d\theta\right\} = \{\frac{\delta(\theta)}{\sqrt{2\pi t\sigma^2}}\left[\exp\left\{-\frac{(y_\nu - x_\nu)^2}{2t\sigma^2}\right\} - \exp\left\{-\frac{(|y_\nu| + |x_\nu|)^2}{2t\sigma^2}\right\}\right] +$$

$$+(1 + q\,\mathrm{sign}\, y_\nu)\frac{\sigma^2\theta + |x_\nu| + |y_\nu|}{\sqrt{2\pi t^3\sigma^2}}\exp\left\{-\frac{(\sigma^2\theta + |x_\nu| + |y_\nu|)^2}{2t\sigma^2}\right\}\}g^S(t,x,y)dyd\theta$$

Remark. $\mathbf{P}_x\left\{\tilde{x}(t) \in dy, \eta_t \in d\theta\right\}$ has the atom at the point $\theta = 0$.

Proof. Consider

$$u(t,x,\lambda,\mu) = \tilde{\mathbf{E}}_x e^{i(\tilde{x}(t),\mu) + i\lambda\eta_t}, t > 0, x \in \Re^d, \lambda \in \Re, \mu \in \Re^d$$

where $\tilde{\mathbf{E}}_x$ is the averaging operator corresponding $\tilde{g}(t,x,y)$.

The function $u(t,x,\lambda,\mu)$ satisfies the following integral equation (see [3])

$$(2)\qquad u(t,x,\lambda,\mu) = \int_{\Re^d} e^{i(y,\mu)}\tilde{g}(t,x,y)dy + i\lambda\int_0^t d\tau\int_S u(t-\tau,z,\lambda,\mu)\tilde{g}(\tau,x,z)dz$$

Note that the function $u(t,\cdot,\lambda,\mu)$ on the right hand side of (2) depends on its values on S. Then using Fourier-Laplace transformation we can find $u(t,x,\lambda,\mu)$ for $x \in S$

$$u(t,x,\lambda,\mu) = \int_0^\infty d\theta\int_{\Re^d} e^{i(y,\mu)+i\lambda\theta}(1 + q\,\mathrm{sign}\, y_\nu)\times$$

$$\times\frac{1}{\sqrt{2\pi t\sigma^2}}\exp\left\{-\frac{(\sigma^2\theta + |x_\nu| + |y_\nu|)^2}{2t\sigma^2}\right\}g^S(t,x,y)dy,$$

Substituting this relation in (2) we will obtain

$$u(t,x,\lambda,\mu) = \int_{\Re^d}\frac{e^{i(y,\mu)}}{\sqrt{2\pi t\sigma^2}}\left[\exp\left\{\frac{-(y_\nu - x_\nu)^2}{2t\sigma^2}\right\} - \exp\left\{\frac{-(|y_\nu| + |x_\nu|)^2}{2t\sigma^2}\right\}\right]g^S(t,x,y)dy +$$

$$+\int_0^\infty d\theta\int_{\Re^d} e^{i(y,\mu)+i\lambda\theta}(1 + q\,\mathrm{sign}\, y_\nu)\frac{\sigma^2\theta + |x_\nu| + |y_\nu|}{\sqrt{2\pi t^3\sigma^2}}\times$$

$$\times\exp\left\{-\frac{(\sigma^2\theta + |x_\nu| + |y_\nu|)^2}{2t\sigma^2}\right\}g^S(t,x,y)dy$$

where $t > 0, x \in \Re^d, \lambda \in \Re, \mu \in \Re^d$. This gives us $\mathbf{P}_x\left\{\tilde{x}(t) \in dy, \eta_t \in d\theta\right\}$.

If we know the joint distribution of $\tilde{x}(t)$ and η_t, then the equation (1) can be solved.

Theorem 2. *The solution of the equation* (1) *is a continuous Markov process* $x(t)$ *with transition probability density* $G(t, x, y)$.

Proof. We have

$$\mathbf{P}_x \{x(t) \in dy\} = \int_{[0,+\infty)} \mathbf{P}_x \{x(t) \in dy, \eta_t \in d\theta\} \tag{3}$$

When $\theta = 0$ then $x(t) = \tilde{x}(t)$ and joint distribution $\mathbf{P}_x \{x(t) \in dy, \eta_t \in d\theta\}$ is equal to $\mathbf{P}_x \{\tilde{x}(t) \in dy, \eta_t \in d\theta\}$ at the point $\theta = 0$. For each $\theta \in (0, \infty)$ the process $x(t)$ is equal to $\tilde{x}(t) + \alpha\theta$. Integrating (3) on θ we obtain that

$$\mathbf{P}_x \{x(t) \in dy\} = \frac{1}{\sqrt{2\pi t\sigma^2}} \left[\exp\left\{ -\frac{(y_\nu - x_\nu)^2}{2t\sigma^2} \right\} - \exp\left\{ -\frac{(|y_\nu| + |x_\nu|)^2}{2t\sigma^2} \right\} \right] g^S(t, x, y)dy$$

$$+ \int_0^\infty (q \operatorname{sign} y_\nu + 1) \frac{\sigma^2\theta + |x_\nu| + |y_\nu|}{\sqrt{2\pi t^3\sigma^2}} \exp\left\{ -\frac{(\sigma^2\theta + |x_\nu| + |y_\nu|)^2}{2t\sigma^2} \right\} g^S(t, x + \alpha\theta, y) d\theta dy$$

It coincides with the result of the Theorem 1.

References

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